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# Necessary conditions for the occurrence of a naked singularity in higher dimensional dust collapse

R V Saraykar and K D Patil*[1]

Department of Mathematics, Nagpur University Campus, Nagpur-440 033, Maharashtra, India

[1]Department of Mathematics, B D College of Engineering, Sewagram, Wardha-442 001, Maharashtra, India

E-mail : kishordpatil@yahoo.com

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**Abstract** : We analyse here higher dimensional non-marginally bound inhomogeneous dust collapse where initial data consists of finitely differentiable functions of comoving coordinate $r$. We show that with the introduction of a velocity distribution function, the results on the occurrence of a naked singularity get slightly modified in comparison with the marginally bound collapse. We also discuss the nature of families of radial null geodesics coming out from the singularity.

**Keywords** : Naked singularity, black hole, gravitational collapse, cosmic censorship hypothesis.

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## 1. Introduction

The final outcome of a continual gravitational collapse is an open problem in classical general relativity. However, it has been proved that under fairly general hypotheses, solutions of Einstein field equations with physically realistic matter can develop into singularities as a result of gravitational collapse [1]. The key problem that still remains unsolved is the nature of such singularities. The main open issue is whether the singularities which arise as the end product of collapse, can actually be observed. According to cosmic censorship hypothesis (CCH) [2,3], all spacetime singularities evolving from physically reasonable initial data are always covered. Many types of gravitational collapse have been studied so far in the context of the CCH [4,5].

Lately, the study of higher dimensional spacetimes has led to important generalizations and wider understanding of solutions of Einstein field equations. Hence, though higher dimensional spacetimes are not so realistic, to study cosmic censorship hypothesis, it now becomes essential to study the gravitational collapse of a matter in the higher dimensional spacetimes. Previously, some papers on higher dimensional dust collapse have appeared [6–11]. In Ref. [7], gravitational collapse of an inhomogeneous dust cloud described by a self-similar higher dimensional Tolman-Bondi solution has been analyzed.

It has been shown in [6] that in a higher dimensional marginally bound dust collapse with $D \geq 6$, naked singularities occur if first non-vanishing derivative of density is $\rho_1$ only (under the assumption that density is higher at the center and it is decreasing away from the center). Considering non-marginally bound dust collapse, it has been argued in [8] that even though first derivative of density $\rho_1$ is zero (for $D \geq 6$), central shell focussing singularity could be naked. Since the closed form of solutions for non-marginally bound dust collapse are known only in five-dimensional spacetimes [9], we analyse this spacetime where the initial data (density and velocity distribution functions) are finitely differential functions of comoving coordinate $r$.

Occurrence and nature of singularities in higher dimensional marginally-bound dust collapse have been discussed in details in Ref [6]. This analysis would not be complete unless one studies the non-marginally bound collapse as well. Thus, it is an interesting question to

*Corresponding Author

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examine whether these results would carry in the non-marginally bound case.

The paper is organized as follows. In Section 2, we give notations and preliminaries about the five dimensional inhomogeneous Tolman-Bondi model. In Section 3 we discuss the occurrence and nature of singularities arising in this spacetime. In the concluding Section 4, we remark on naked singularities in ($N$+2)-dimensional non marginally bound spacetimes.

## 2. Tolman-Bondi solution in five-dimensional spacetime

The line element of a spherically symmetric inhomogeneous dust cloud in five-dimensional spacetime is given by [10]

$$ds^2 = -dt^2 + \frac{R'^2}{1+f(r)} dr^2 +$$

$$R^2\left(d\theta_1{}^2 + \sin^2\theta_1 d\theta_2{}^2 + \sin^2\theta_1 \sin^2\theta_2 d\theta_3{}^2\right), \quad (1)$$

where $f(r)$ is an arbitrary function of comoving coordinate $r$, satisfying $f > -1$. $R(t,r)$ is the physical radius at a time $t$ of the shell lebelled by $r$. It has been argued in [12] that for $f(r) > 0$, collapse is not essentially gravitational but due to some initial conditions infinitely far in the past, so we study the case when $f(r) < 0$.

The energy momentum tensor is given by

$$t^{ij} = \varepsilon \delta^i_t \delta^j_t, \quad (2)$$

where

$$\varepsilon(t,r) = \frac{3F'}{2R^3 R'}. \quad (3)$$

We assume energy density $\varepsilon(t,r)$ such that it is higher at the center and decreasing away from the center.

The function $R(r,t)$ is given by

$$\dot{R}^2 = \frac{F(r)}{R^2} + f(r). \quad (4)$$

Here, the dot and prime denote partial derivatives with respect to $t$ and $r$, respectively.

Since we are considering the collapsing model, it require $\dot{R} < 0$.

Integration of eq. (4) yields the solution [13]

$$t - t_s(r) = \frac{-R^2}{\sqrt{F}} G\left(\frac{fR^2}{F}\right), \quad (5)$$

where $G(y)$ is a strictly real positive and bounded function and is given by

$$G(y) = \sqrt{\frac{1+y}{y}}, \quad y \neq 0;$$

$$= \frac{1}{2}, \quad y = 0. \quad (6)$$

Using scaling freedom $R(0,r) = r$, eq. (5) gives

$$t_s(r) = \frac{r^2}{\sqrt{F}} G\left(\frac{r^2 f}{F}\right) \quad (7)$$

where $t_s(r)$ gives the time at which the physical radius ($R$) becomes zero, hence ranges for $t$ and $r$ are given by

$$-\infty < t < t_s(r) \quad \text{and} \quad 0 \le r < \infty .$$

Since the shell-crossing singularities ($R' = 0$, $R > 0$) are gravitationally weak [12] we consider only the shell-focussing singularities ($R = 0$).

The function $F(r)$ and $f(r)$ can be determined from the initial density $\rho(r) = \varepsilon(r,0)$ and the initial velocity $V(r) = R'(r,0)$.

From eq. (3) it follows that

$$F(r) = \frac{2}{3}\int \rho(r) r^3 dr . \quad (8)$$

We assume that initial density profile $\rho(r)$ has the series expansion [14] :

$$\rho(r) = \rho_0 + \rho_1 r + \frac{\rho_2 r^2}{2!} + \frac{\rho_3 r^3}{3!} + \frac{\rho_n r^n}{n!} + \cdot \quad (9)$$

near the center $r = 0$, which can be substituted in the eq. (8) to yield

$$F = F_0 r^4 + F_1 r^5 + F_2 r^6 + F_n r^{n+4} + \cdots, \quad (10)$$

where

$$F_n = \frac{2}{3} \frac{\rho_n}{n!(N+4)}, \quad (11)$$

$\rho_n$ being the $n$-th derivative of density at the center.

From eq. (4) one may write

$$f(r) = V^2(r) - \frac{F(r)}{r^2}. \quad (12)$$

It follows from eqs. (10) and (12) that the expansion of $f(r)$ should begin with a term that is of order $r^2$ or higher.

So we expand $f(r)$ as

$$f(r) = f_2 r^2 + f_3 r^3 + K + f_n r^n + \cdots \tag{13}$$

and assume $f_2 \neq 0$.

Though the solution for non-marginally bound collapse in 4D case is parametric form and it is in 5D case is in the closed form, it would be interesting to see that the notations in 5D case can be written in a similar fashion to that of 4D case [4] with slight difference.

Thus using eqs. (3)–(7), we can write

$$R' = r^{\alpha-1} \frac{1}{2}(\eta-\beta)X + |\Theta - | \frac{\eta}{2} - \beta$$

$$X^2 G(PX^2)\Big]\left[P + \frac{1}{X^2}\right]^{1/2} \tag{14}$$

$$= r^{\alpha-1} H(X,r). \tag{15}$$

where we have used the following notations :

$$u = r^{\alpha},\ X = R/u,\ \eta(r) = \frac{rF'}{F},\ \beta(r) = \frac{rf'}{f},$$

$$p(r) = \frac{r^2 f}{F} \quad A = \frac{\sqrt{F}}{u},\ P = \frac{fu^2}{F}$$

$$\frac{1}{r^{2(\alpha-1)}}\ G(p)\left(\frac{\eta}{2} - \beta\right) + \frac{2+\beta-\eta}{2\sqrt{1+p}} \tag{16}$$

$$H(X,r) = \frac{1}{2}(\eta-\beta)X + \left[\Theta - \left(\frac{\eta}{2} - \beta\right)X^2 G(PX^2)\right.$$

$$P + \frac{1}{\cdot}^{1/2} \tag{17}$$

Kretschmann scalar ($K = R_{abcd}R^{abcd}$) for the metric (1) is given by

$$K = \frac{AF'^2}{R^6 R'^2} + \frac{BFF'}{R^7 R'} + \frac{CF^2}{R^8}, \tag{18}$$

where $A$, $B$, $C$ are some constants.

Outgoing radial null geodesics for the metric (1) is given by

$$\frac{dt}{dr} = \frac{R'}{\sqrt{1+f(r)}}. \tag{19}$$

Writing above equation in terms of variables ($u = r^{\alpha}$, $R$) where $\alpha > 1$, we have

$$\frac{dR}{du} = \frac{H(X,u)}{\alpha}\ 1 - \sqrt{(\Lambda^2/X^2)+f} = U(X,u). \tag{20}$$

Let us consider the limit $X_0$ of the tangent $X$ along the null geodesic terminating at the singularity at $R = 0$, $u = 0$.

Thus,

$$X_0 = \lim_{\substack{R\to 0\\ u\to 0}} \frac{R}{u} = \lim_{\substack{R\to 0\\ u\to 0}} \frac{dR}{du} = \lim_{\substack{R\to 0\\ u\to 0}} U(X,u). \tag{21}$$

If a real positive value of $X_0$ satisfies the above equation then the singularity would be naked. If the singularity is naked, there exists some $\alpha$ such that at least one finite positive value of $X_0$ exists which solves the algebraic equation

$$V(X_0) = 0, \tag{22}$$

where $V(X) = U(X,0) - X$

$$\frac{H(X,0)}{\alpha}\ 1 - \frac{\sqrt{(\Lambda_0^2/X^2)+f_0}}{\sqrt{1+f_0}} - X = 0. \tag{23}$$

The value of $\alpha$ should be chosen in such a way that $\Theta_0$ should not be equal to zero or infinity.

## 3. Naked singularities in non-marginally bound collapse

To study the nature of singularity, we determine the quantities defined in the eqs. (15)–(17).

We find that

$$p_0 = \lim_{r\to 0} \frac{r^2 f}{F} = \frac{3f_2}{\rho_0}. \tag{24}$$

Expression for $\Theta_0$ then becomes

$$\Theta_0 = \left| 1 + \frac{3f_2}{\rho_0}\right)^{-1/2} \lim_{r\to 0} \frac{\left[2 + \frac{rf'}{f} - \frac{rF'}{F}\right]}{2r^{2(\alpha-1)}} +$$

$$\lim_{r\to 0} \frac{\left(\frac{rF'}{F} - \frac{2rf'}{f}\right) G\left(\frac{3f_2}{\rho_0}\right)}{2r^{2(\alpha-1)}}. \tag{25}$$

Let

$$g(r) = 2 + \frac{rf'}{f} - \frac{rF'}{F} \tag{26}$$

and

$$h(r) = \frac{rF'}{F} - \frac{2rf'}{f}. \tag{27}$$

Since $\eta_0 = 4$, $\beta_0 = 2$, it follows that

$$\lim_{r\to 0} g(r) = \lim_{r\to 0} h(r) = 0. \tag{28}$$

Hence for $\alpha > 1$, we have

$$\lim_{r\to 0} \frac{g(r)}{2r^{2(\alpha-1)}} = \lim_{r\to 0} \frac{g'(r)}{4(\alpha-1)r^{2\alpha-3}}, \tag{29}$$

and

$$\lim_{r\to 0} \frac{h(r)}{2r^{2(\alpha-1)}} = \lim_{r\to 0} \frac{h'(r)}{4(\alpha-1)r^{2\alpha-3}} \tag{30}$$

Now, we consider different cases for the initial data distribution.

*Case 1:* $f_3 \neq 0$ or $\rho_1 \neq 0$.

Since

$$\lim_{r\to 0} g'(r) = \frac{f_3}{3f_2} - \frac{4\rho_1}{5\rho_0}, \tag{31}$$

and

$$\lim_{r\to 0} h'(r) = \frac{4\rho_1}{5\rho_0} - \frac{2f_3}{3f_2}, \tag{32}$$

it follows that in case either

$$\frac{f_3}{3f_2} - \frac{4\rho_1}{5\rho_0} \neq 0, \tag{33}$$

or

$$\frac{4\rho_1}{5\rho_0} - \frac{2f_3}{3f_2} \neq 0, \tag{34}$$

we can choose $\alpha = 3/2$ to get

$$\Theta_0 = \left(1 + \frac{3f_2}{\rho_0}\right)^{-1/2} \left[\frac{f_3}{6f_2} - \frac{2\rho_1}{5\rho_0}\right] + G\left(\frac{3f_2}{\rho_0}\right)\left[\frac{2\rho_1}{5\rho_0} - \frac{f_3}{3f_2}\right]. \tag{35}$$

The condition (33) or (34) is satisfied if at least one of $f_3$ and $\rho_1$ is non-zero, and in that case, $\Theta_0$ will be finite and non-zero.

Thus for $\alpha = 3/2$, we have

$$P_0 = 0,\ \Lambda_0 = 0,\ H(X_0, 0) = X_0 + \frac{\Theta_0}{X_0}. \tag{36}$$

Eq. (23) then gives

$$X_0^2 = 2\Theta_0, \tag{37}$$

since $\Theta_0$ has finite and positive value, $X_0$ must be positive and real, which shows that singularity arising in this case is naked.

*Case 2 :* $(\rho_1, f_3) = 0$ and $\rho_2 \neq 0$ or $f_4 \neq 0$.

If $\rho_1$, $f_3$ both are zero, then $\Theta_0$ will be zero and hence we apply L-Hospital's rule to eqs. (29) and (30) again. In this case, we choose $\alpha > 3/2$ and therefore,

$$\lim_{r\to 0} \frac{g(r)}{2r^{2(\alpha-1)}} = \lim_{r\to 0} \frac{g''(r)}{4(\alpha-1)(2\alpha-3)r^{2\alpha-4}} \tag{38}$$

and

$$\lim_{r\to 0} \frac{h(r)}{2r^{2(\alpha-1)}} = \lim_{r\to 0} \frac{h''(r)}{4(\alpha-1)(2\alpha-3)r^{2\alpha-4}} \tag{39}$$

Since

$$\lim_{r\to 0} g''(r) = \frac{f_4}{3f_2} - \frac{4\rho_2}{3\rho_0} \neq 0, \tag{40}$$

and

$$\lim_{r\to 0} g''(r) = \frac{f_4}{3f_2} - \frac{4\rho_2}{3\rho_0} \neq 0, \tag{41}$$

in order to get a finite and non-zero value of $\Theta_0$, we choose $\alpha = 2$.

Hence,

$$= \left[1 + \frac{3f_2}{\rho_0}\right]^{-1/2} \left[\frac{f_4}{12f_2} - \frac{\rho_2}{3\rho_0}\right] + G\left(\frac{3f_2}{\rho_0}\right)\left[\frac{\rho_2}{3\rho_0} - \frac{f_4}{6f_2}\right] \tag{42}$$

Thus for $\alpha = 2$, we find $P_0 = 0$, $\Lambda_0 = \sqrt{F_0}$.

Eq. (23) then becomes

$$1 - \frac{\sqrt{F_0}}{\phantom{0}}\left[X_0 + \frac{\Theta_0}{X_0}\right] - X_0 = 0. \tag{43}$$

Defining $y = X/\sqrt{F_0}$ and $\xi = -\Theta_0/F_0$, above equation can be expressed as

$$2y^3 + 2y^2 + y\xi - \xi = 0. \tag{44}$$

Singularity could be naked if eq. (44) has at least one positive and real root.

Numerical calculations show that eq. (44) has real and positive root (in fact two) if

$$\xi \le \frac{1-\sqrt{5}}{9-4\sqrt{5}} \quad i.e.\ \xi \le -22.18033, \tag{45}$$

which is the same condition obtained for marginally-bound collapse [6].

Thus, the central shell-focusing singularity is naked if the condition (45) is satisfied, otherwise collapse ends in a black hole.

In the case discussed above, one interesting situation arises when both $\rho_1 = \rho_2 = 0$. We know that in this situation, marginally-bound collapse always ends into a black hole [6]. But it can be observed from eq. (42) that even if we have initial density $\rho_{(r)}$ such that $\rho_1 = \rho_2 = 0$, we can choose velocity distribution function $f(r)$ such that (choosing $f_4 \neq 0$) the singularity is either naked or covered.

*Case 3 :* $(\rho_1, f_3) = (\rho_2, f_4) = 0$

In this case we have to choose $\alpha$ greater than 2 which makes $\Lambda_0 = \infty$ and hence eq. (23) could not have positive root and so collapse ends into a black hole.

We have discussed the existence of a real and positive root to the root eq. (23). Now, we investigate whether a family of outgoing radial null geodesics can terminate at the singularity with this root as a tangent. To check this, we may apply the method described in [4]. This matter has been analyzed in details in the case of marginally-bound space time in Ref. [6]. Since the calculations for the non-marginally bound case are also being similar to that of marginally bound case, to avoid repetitions, we give only the results. Thus following the calculations in Ref. [6], it can be observed that for the first case (*i.e.* $\rho_1 \neq 0$ or $f_3 \neq 0$), there is only single null geodesic coming out from the singularity having $X_0$ as a tangent. For the second case (*i.e.* for $f_3 = \rho_1 = 0$ and $f_4 \neq 0$ or $\rho_2 \neq 0$), it can be checked that out of the two positive roots, there are infinite families of radial null geodesics coming out along smaller root whereas there will be only single radial null geodesic coming out along larger root.

## 4. Concluding remarks

Together with the work in Ref. [14], we observe that in 4D case, leading three derivatives of density and leading five derivatives of velocity distribution function decide the nature of a singularity. While in 5D case, first two derivatives of density and first four derivatives of velocity distribution function play this role. Also it has been argued in [8] that in the higher dimensional non-marginally bound collapse with $D \ge 6$, even the condition $\rho_1 = 0$ does not save the CCH. Hence taking above results into consideration, one may argue that as the dimension of a spacetime increases, the calculations of derivatives of both density as well as velocity distribution functions of less orders near the center are required to decide the nature of singularity.

It has been shown in Ref. [6] that in a higher dimensional marginally-bound dust collapse with $D \ge 6$, there is only one outgoing null geodesic coming out from the singularity having $X_0$ as a tangent. Hence, though the closed form of solutions for a non-marginally bound collapse in six or higher-dimensional spacetimes are not known, we may argue that the above result is true in the case of non-marginally bound collapse as well.

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